

PRODUCTS OF CONJUGACY CLASSES OF THE ALTERNATING GROUP

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ABSTRACT. Let A_n be the alternating group on n letters. For $n > 5$, we describe the elements $\alpha, \beta \in A_n$ when $\alpha^{A_n}\beta^{A_n}$ is the union of at most four distinct conjugacy classes.

1. INTRODUCTION

Let G be a finite group. Then the product of two conjugacy classes a^G, b^G in G is the union of m distinct conjugacy classes of G , for some integer $m > 0$. We set $\eta(a^G b^G) = m$. In this note, we continue our exploration of the minimum possible value of $\eta(\alpha^G \beta^G)$, begun in [1], investigating the case where G is the symmetric or alternating group on n letters.

Throughout this paper we denote by $\min(n)$ the smallest integer in the set $\{\eta(\alpha^{A_n} \beta^{A_n}) \mid \alpha, \beta \in A_n \setminus \{e\}\}$. It is known that for $n \geq 5$, the product of two conjugacy classes of A_n is never a conjugacy class, so for α, β nontrivial, $\eta(\alpha^{A_n} \beta^{A_n}) \geq 2$. On the other hand, for any $n > 5$, we can check that $\eta((1\ 2\ 3)^{A_n} (1\ 2\ 3)^{A_n}) = 5$, so $2 \leq \min(n) \leq 5$. The main result of this paper is the following theorem.

Theorem 1. *Fix an integer $n \geq 6$. Let A_n be the alternating group on n letters and $\alpha, \beta \in A_n$ be nontrivial elements. Assume that $\eta(\alpha^{A_n} \beta^{A_n}) < 5$. Then either α or β is a product of transpositions. Assume that α is a product of transpositions. Then one of the following holds:*

- i) n is a multiple of 4, α is the product of $\frac{n}{2}$ disjoint 2-cycles and β is a 3-cycle, and $\eta(\alpha^{A_n} \beta^{A_n}) = 2$.*
- ii) $n - 1$ is a multiple of 4, α is the product of $\frac{n-1}{2}$ disjoint 2-cycles and β is a 3-cycle, and $\eta(\alpha^{A_n} \beta^{A_n}) = 4$.*

Since for any group G and any $a, b \in G$, we have that $a^G b^G = b^G a^G$, the previous result describes all the possible $\alpha, \beta \in A_n \setminus \{e\}$ such that $\eta(\alpha^{A_n} \beta^{A_n}) < 5$.

Corollary 1. *For $n \geq 6$,*

$$\min(n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{4} \\ 4 & \text{if } n \equiv 1 \pmod{4} \\ 5 & \text{otherwise} \end{cases}$$

Remark 1. Values of $\min(n)$ for $n \leq 12$ can be computed using a computer program such as GAP [3], and are given in the tables in the Appendix.

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Remark 2. For most pairs $\alpha, \beta \in A_n$, $\eta(\alpha^{A_n} \beta^{A_n})$ is much larger than $\min(n)$ as shown in the tables in the Appendix.

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2. NOTATION

We use the standard cycle notation for permutations, and we take our maps to be written on the right, so we multiply cycles from left to right. Thus for example,

$$(1\ 2\ 4)(1\ 2\ 3)(4\ 5\ 6) = (1\ 3)(2\ 5\ 6\ 4).$$

For integers $n > m$, any element of A_m can also be considered as an element of A_n , so we use this fact without comment.

The following is a very well known result.

Lemma 2. *Let $\alpha \in S_n$. Then*

- a) α can be written as a product of disjoint cycles.*
- b) α^{S_n} is the set of all permutations of S_n with the same cycle structure as α .*

Definition 3. The cycle type of a element of S_n is an unordered multiset of integers, forming a partition of n . We use square bracket notation, for example, For $\alpha \in S_n$, the cycle type of α can be written as

$$(2.1) \quad [\alpha] = [\underbrace{1, \dots, 1}_{n_0}, \underbrace{a_1, \dots, a_1}_{n_1}, \underbrace{a_2, \dots, a_2}_{n_2}, \dots, \underbrace{a_r, \dots, a_r}_{n_r}],$$

where α consists of disjoint cycles of lengths a_1, \dots, a_r ,

$$\begin{aligned} r &\geq 0, \\ n_i &\geq 1 \text{ for } 1 \leq i \leq r, \text{ and } n_0 \geq 0 \\ 1 &< a_1 < a_2 < \dots < a_{r-1} < a_r, \\ n_0 + \sum_{i=1}^r n_i a_i &= n. \end{aligned}$$

i.e., here the cycle lengths are written in increasing order of size, and form a partition of n .

Example 1. For the following element of A_{12} we have

$$[(8\ 9)(5\ 6\ 7\ 1)(11)(12)] = [1, 1, 1, 1, 1, 1, 2, 4]$$

Lemma 2 says that two elements γ, δ in S_n are conjugate if and only if they are of the same type, that is if and only if $[\gamma] = [\delta]$.

3. PROOFS

3.1. Reducing from A_n to S_n . Although our main result is about conjugacy classes in A_n , by means of the following results, we will reduce most of the work to determining pairs $\alpha, \beta \in S_n$ for which $\eta(\alpha^{S_n} \beta^{S_n}) < 5$. Note that for $\alpha \in A_n$, if $\alpha^{A_n} \neq \alpha^{S_n}$, then α^{S_n} is the union of two A_n conjugacy classes.

We will make use of the following two well known lemmas; the proofs are included for completeness.

Lemma 4. *Let $\alpha \in A_n$ and $\beta \in S_n \setminus A_n$. Assume that $\alpha^\beta = \beta^{-1} \alpha \beta = \alpha$. Then all the permutations of type $[\alpha]$ are A_n -conjugate to α , i.e., $\alpha^{A_n} = \alpha^{S_n}$. Conversely, suppose that $\alpha^{A_n} = \alpha^{S_n}$, then $\alpha = \alpha^\beta$ for some $\beta \in S_n \setminus A_n$.*

Proof. First assume that $\alpha = \alpha^\beta$ for some odd β . Let γ be any permutation with $[\gamma] = [\alpha]$. By Lemma 2 there exists $\delta \in S_n$ such that $\delta^{-1}\alpha\delta = \gamma$. If $\delta \in A_n$, then γ and α are A_n -conjugates. Otherwise $\beta\delta \in A_n$. Observe that $(\beta\delta)^{-1}\alpha(\beta\delta) = \gamma$ and thus α and γ are A_n -conjugates.

Suppose that $\alpha^{A_n} = \alpha^{S_n}$. So for some element $\gamma \in S_n \setminus A_n$, and some $\delta \in A_n$, we have $\alpha^\gamma = \alpha^\delta$. So $\alpha^{\gamma\delta^{-1}} = \alpha$, so we can take $\beta = \gamma\delta^{-1}$. \square

Corollary 5. *Let $\alpha \in A_n$. If for some integer $m > 0$ and some permutation $\alpha_1 \in A_{n-2m}$ the cycle type of α differs from that of α_1 by insertion of either $\{2m\}$ or $\{m, m\}$, then $\alpha^{A_n} = \alpha^{S_n}$.*

Proof. If $\alpha = \alpha_1\gamma$, where γ is a cycle of order $2m$ and γ and α_1 are disjoint permutations, then $\alpha^\gamma = \alpha$. Since a cycle of even order is an odd permutation, the result follows from Lemma 4, since $\alpha^\gamma = \alpha$.

Assume now that $\alpha = \gamma_1\gamma_2\alpha_1$, where γ_1 and γ_2 are cycles of order m and α_1 , γ_1 and γ_2 are disjoint permutations. If m is even, the result then follows by the previous paragraph. We may assume then that m is odd. Without loss of generality, we may assume that $\gamma_1\gamma_2 = (1\ 2\ \dots\ m)(m+1\ m+2\ \dots\ 2m)$. Observe that the permutation $\delta = (1\ m+1)(2\ m+2)\dots(i\ m+i)\dots(m\ 2m)$ is an odd permutation and $(\gamma_1\gamma_2)^\delta = \gamma_2\gamma_1$. Thus $\alpha^\delta = \alpha$ and the result follows. \square

Lemma 6. *Let $\alpha \in A_n$. Set $[\alpha] = [a_1, \dots, a_r]$. Then α^{S_n} is the union of two distinct conjugacy classes of A_n (equivalently, $\alpha^{S_n} \neq \alpha^{A_n}$) if and only if a_i is an odd integer for all i and $a_i \neq a_j$ for any $i \neq j$.*

Proof. By Lemma 4, notice that if some a_i is even for some i , then all the elements of type $[a_1, \dots, a_r]$ are A_n -conjugate, since we can just conjugate by the corresponding cycle of length a_i . If there exist a_i and a_j such that $a_i = a_j$, then α will be fixed by an odd permutation, as in the second case in Corollary 5.

Conversely, assume that a_i is an odd integer for all i and $a_i \neq a_j$ for any $i \neq j$. By renaming if necessary, we may assume that $\alpha = (1\ 2\ \dots\ a_1)\alpha_1$, with α_1 fixing $1, 2, \dots, a_1$. Let $\beta = \alpha^{(1\ 2)} = (2\ 1\ 3\ \dots\ a_1)\alpha_1$. We can check that since $a_i \neq a_j$ and a_i is odd for all i , if $\alpha^\gamma = \beta$, then $\gamma = (1\ 2)$. Thus α and β are not A_n conjugates but clearly $[\alpha] = [\beta]$. \square

Corollary 7. *Let $\alpha \in A_n$. Suppose that $\alpha^{S_n} \neq \alpha^{A_n}$. Then α fixes at most one element. If $\alpha = (1\ 2\ 3\ \dots\ a_1)\alpha_1$, where α_1 fixes $1, 2, \dots, a_1$, then $\alpha^{S_n} = \alpha^{A_n} \cup (\alpha^{(1\ 2)})^{A_n} = \alpha^{A_n} \cup ((2\ 1\ 3\ \dots\ a_1)\alpha_1)^{A_n}$.*

Proposition 8. *Let $\alpha, \beta \in A_n$, and suppose that either $\alpha^{A_n} = \alpha^{S_n}$ or $\beta^{A_n} = \beta^{S_n}$. Then $\eta(\alpha^{A_n}\beta^{A_n}) \geq \eta(\alpha^{S_n}\beta^{S_n})$.*

Proof. Suppose that $\alpha^{A_n} = \alpha^{S_n}$. Then by Lemma 4 there is some odd permutation γ with $\alpha^\gamma = \alpha$. For any $a, b \in S_n$, one of the pairs (a, b) , $(\gamma a, b)$, $(\gamma a\gamma^{-1}, b\gamma^{-1})$, $(a\gamma^{-1}, b\gamma^{-1})$, is in $A_n \times A_n$. Since $\alpha^\gamma = \alpha$, we have $[\alpha^a\beta^b] = [\alpha^{\gamma a}\beta^b] = [(\alpha^{\gamma a}\beta^b)^{\gamma^{-1}}] = [\alpha^{\gamma a\gamma^{-1}}\beta^{b\gamma^{-1}}]$ and $[\alpha^a\beta^b] = [(\alpha^a\beta^b)^{\gamma^{-1}}][\alpha^{\gamma a\gamma^{-1}}\beta^{b\gamma^{-1}}]$, so $\alpha^{A_n}\beta^{A_n}$ contains elements of all possible cycle types of elements of $\alpha^{S_n}\beta^{S_n}$, so the result follows. Similarly if $\beta^{A_n} = \beta^{S_n}$. \square

Remark 3. Experimentally, it seems that $\eta(\alpha^{A_n}\beta^{A_n}) \geq \eta(\alpha^{S_n}\beta^{S_n})$ even when $\alpha^{A_n} \neq \alpha^{S_n}$ and $\beta^{A_n} \neq \beta^{S_n}$, but we have not been able to prove this.

In Proposition 8, we have to assume either $\alpha^{A_n} = \alpha^{S_n}$ or $\beta^{A_n} = \beta^{S_n}$. We now deal with the case where $\alpha^{A_n} \neq \alpha^{S_n}$ and $\beta^{A_n} \neq \beta^{S_n}$.

Lemma 9. *Let G be a finite group, a and b in G . Then $a^G b^G = b^G a^G$.*

Lemma 10. *Let $\alpha, \beta \in A_n$ with $n \geq 7$. Assume that*

$$\begin{aligned}\alpha &= (1\ 2\ 3\ 4\ 5\ 6 \cdots a_1)\gamma, \\ \beta &= (1\ 2\ 3\ 4\ 5\ 6 \cdots b_1)\delta,\end{aligned}$$

with $a_1 \geq 7$ and γ fixing $1, 2, \dots, a_1$, and with $b_1 \geq 7$ and δ fixing $1, 2, \dots, b_1$. Then $\eta(\alpha^{A_n} \beta^{A_n}) \geq 5$ and $\eta(\alpha^{A_n} \beta^{(2\ 4)(3\ 6\ 5)A_n}) \geq 5$

Proof. Let

$$\begin{aligned}\alpha_2 &= (1\ 3\ 2\ 7\ 5\ 6\ 4 \cdots a_1)\gamma, \\ \alpha_3 &= (1\ 3\ 2\ 5\ 4\ 6\ 7 \cdots a_1)\gamma, \\ \alpha_4 &= (1\ 5\ 4\ 3\ 2\ 6\ 7 \cdots a_1)\gamma, \\ \alpha_5 &= (1\ 6\ 5\ 4\ 3\ 2\ 7 \cdots a_1)\gamma.\end{aligned}$$

Then $\alpha, \alpha_1, \alpha_2, \alpha_3$ and α_4 are A_n -conjugates. We will now show that $\alpha\beta, \alpha_2\beta, \alpha_3\beta, \alpha_4\beta$ fix different number of points, and thus are not conjugate to each other.

Let X be the set of fixed points of $\alpha\beta$. Observe that $\{1, 2, 3, 4, 5\} \cap X = \emptyset$. We can check that the set of fixed points of $\alpha_2\beta, \alpha_3\beta, \alpha_4\beta$ and $\alpha_5\beta$ is $X \cup \{3\}, X \cup \{3, 4\}, X \cup \{2, 3, 4\}$, and $X \cup \{2, 3, 4, 5\}$ respectively.

Our proof is similar when β is replaced by

$$\beta' := \beta^{(2\ 4)(3\ 6\ 5)} = (1\ 4\ 6\ 2\ 3\ 5\ 7 \cdots b_1)\delta.$$

Define a further conjugate of α in A_n by

$$\alpha_6 = \alpha^{(2\ 7\ 4\ 3\ 5)} = (1\ 7\ 5\ 3\ 2\ 6\ 4 \cdots a_1)\gamma.$$

Now if Y is the set of fixed points of $\alpha\beta'$, then the sets of fixed points of $\alpha_2\beta', \alpha_3\beta', \alpha_4\beta', \alpha_5\beta', \alpha_6\beta'$ are $Y \cup \{3, 6, 7\}, Y \cup \{3\}, Y \cup \{2, 3\}, Y \cup \{3\}, Y \cup \{2, 3, 6, 7\}$ respectively. The numbers of fixed points distinguishes the conjugacy classes of $\alpha\beta', \alpha_2\beta', \alpha_3\beta', \alpha_4\beta'$ and $\alpha_6\beta'$ from each other. \square

Lemma 11. *Let $\alpha, \beta \in A_n$ with $n \geq 6$. Assume that $\alpha^{A_n} \neq \alpha^{S_n}$ and $\beta^{A_n} \neq \beta^{S_n}$. Then $\eta(\alpha^{A_n} \beta^{A_n}) \geq 5$.*

Proof. By Lemma 6 and Corollary 7, we have that both α and β fix at most one element and they are products of disjoint cycles of different odd order.

By Lemma 6 and computations with GAP, we may assume that both α and β have cycles of length at least 7. This follows, since by Lemma 6, we know that α and β must have cycles of distinct odd lengths. If there is no cycle of length at least 7, then $n = 1, 4, 8$ or 9 . However, we assume $n \geq 6$, so we must have $n = 8$ or 9 , and the cycle types of α and β must be either $[1, 5]$, $[3, 5]$, or $[1, 3, 5]$. From Tables 10, 12 and 13 we see that in these cases, we have $\eta(\alpha^{A_n} \beta^{A_n}) = 7, 13$ and 17 respectively. and so we already have $\eta(\alpha^{A_n} \beta^{A_n}) \geq 5$ in these cases.

By Lemma 7 and Lemma 9, and renaming if necessarily, we may assume that

$$(3.1) \quad \alpha = (1\ 2\ 3\ 4\ 5\ 6 \cdots a_1)\gamma,$$

and

$$(3.2) \quad \text{either } \beta = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \cdots b_1)\delta \text{ or } \beta = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \cdots b_1)^\mu\delta,$$

for some permutations γ and δ , where $a_1 \geq 7$, $b_1 \geq 7$ and γ fixes $1, \dots, a_1$ and δ fixes $1, \dots, b_1$, and μ is any odd permutation. These two cases are dealt with by Lemma 10. Note that by conjugation in A_n we can take any fixed choice of odd μ , and a convenient choice is $\mu = (2 \ 4)(3 \ 6 \ 5)$. \square

3.2. An inductive argument. We would like to apply an inductive argument in this section. We'd like to be able to pass from $\alpha^{S_n}\beta^{S_n}$ to something akin to $\alpha^{S_{n-1}}\beta^{S_{n-1}}$. Instead of using η , we will use η' defined below. To define this we need to choose a particular element, $\mathfrak{s}(\alpha)$ of any conjugacy class α^{S_n} of S_n . Note that η' and \mathfrak{s} are only used in this subsection.

Definition 12. Suppose that $\alpha \in S_n$ has cycle structure as in (2.1). We define the element $\mathfrak{s}(\alpha)$ of α^{S_n} by

$$\mathfrak{s}(\alpha) = (1 + n_0 \ \dots \ a_1 + n_0)(a_1 + n_0 + 1 \ \dots \ 2a_1 + n_0) \cdots (n - a_r + 1 \ \dots \ n).$$

I.e., the cycles are written in increasing order of length, and the elements appearing are written in increasing consecutive order. Elements $1, \dots, n_0$ are fixed. The element n is in a cycle of maximum length, so unless $\alpha = e$, $\mathfrak{s}(\alpha)$ never fixes n .

Example 2. For the following element of A_{12} we have

$$\mathfrak{s}(8 \ 9)(5 \ 6 \ 7 \ 1)(11)(12) = (1)(2)(3)(4)(5)(6)(7 \ 8)(9 \ 10 \ 11 \ 12).$$

Definition 13. Let S_n be the group of permutations on the set $\{1, \dots, n\}$. Define $\eta'(a, b)$ for $a, b \in S_n$ by

$$\eta'(a, b) = \#\{[(\mathfrak{s}(a)^{-1})^\sigma \mathfrak{s}(b)] : \sigma \in S_n, \sigma \text{ fixes } n\}.$$

This definition is chosen because in the inductive step, (details in Proposition 15), as well as having $(n)\sigma = n$, we will also take $(n-1)\sigma = (n-1)$. In this case, provided both α and β are not the identity, we have $(n-1)\mathfrak{s}(\alpha) = n$, $(n-1)\mathfrak{s}(\beta) = n$, and $(n)\mathfrak{s}(\alpha)^{-1} = n-1$, so

$$(3.3) \quad \begin{aligned} (n)(\mathfrak{s}(\alpha)^{-1})^\sigma \mathfrak{s}(\beta) &= (n)\sigma^{-1} \mathfrak{s}(\alpha)^{-1} \sigma \beta = (n)\mathfrak{s}(\alpha)^{-1} \sigma \beta \\ &= (n-1)\sigma^{-1} \mathfrak{s}(\beta) = (n-1)\mathfrak{s}(\beta) = n. \end{aligned}$$

Lemma 14. With $a, b \in S_n$

$$\eta(a^{S_n}b^{S_n}) \geq \eta'(a, b).$$

Proof. For any $\beta \in S_n$, β is conjugate to β^{-1} and to $s(\beta)$. So since $\alpha^\sigma \beta^\tau$ is conjugate to $\alpha^{\sigma\tau^{-1}}\beta$, and since α^{S_n} depends on α only up to conjugacy, we have

$$\eta(a^{S_n}b^{S_n}) = \#\{[(\mathfrak{s}(a)^{-1})^\sigma \mathfrak{s}(b)] : \sigma \in S_n\}.$$

The set involved contains the set used to define η' , and so the inequality follows. \square

Thus in order to find a lower bound on $\eta(a^{S_n}b^{S_n})$, it suffices to find a lower bound on $\eta'(a, b)$.

Proposition 15. *Fix an integer $n > 1$. Let $\alpha, \beta \in S_n \setminus e$ with cycle structure*

$$\begin{aligned} [\alpha] &= \underbrace{[1, \dots, 1]}_{n_0}, \underbrace{[a_1, \dots, a_1]}_{n_1}, \underbrace{[a_2, \dots, a_2]}_{n_2}, \dots, \underbrace{[a_r, \dots, a_r]}_{n_r} \\ [\beta] &= \underbrace{[1, \dots, 1]}_{m_0}, \underbrace{[b_1, \dots, b_1]}_{m_1}, \underbrace{[b_2, \dots, b_2]}_{m_2}, \dots, \underbrace{[b_s, \dots, b_s]}_{m_s} \end{aligned}$$

where

$$\begin{aligned} (3.4) \quad & r, s \geq 1, n_i, m_j \geq 1, \\ & n_i \geq 1 \text{ for } 1 \leq i \leq r, m_i \geq 1 \text{ for } 1 \leq i \leq s, \\ & 1 < a_1 < a_2 < \dots < a_{r-1} < a_r, 1 < b_1 < b_2 < \dots < b_{s-1} < b_s, \\ & \sum_{i=1}^r n_i a_i = \sum_{i=1}^s m_i b_i = n, \\ & \alpha(2) = 1, \beta(1) = 2, \\ & \alpha^{a_1}(2) = \beta^{a_1}(2) = 2. \end{aligned}$$

Then there exist $\alpha', \beta' \in S_{n-1}$ with cycle structure

$$\begin{aligned} [\alpha'] &= \underbrace{[1, \dots, 1]}_{n_0}, \underbrace{[a_1, \dots, a_1]}_{n_1}, \underbrace{[a_2, \dots, a_2]}_{n_2}, \dots, \underbrace{[a_r - 1, a_r, \dots, a_r]}_{+1}, \underbrace{[a_r, \dots, a_r]}_{n_r - 1} \\ [\beta'] &= \underbrace{[1, \dots, 1]}_{m_0}, \underbrace{[b_1, \dots, b_1]}_{m_1}, \underbrace{[b_2, \dots, b_2]}_{m_2}, \dots, \underbrace{[b_s - 1, b_s, \dots, b_s]}_{+1}, \underbrace{[b_s, \dots, b_s]}_{m_s - 1} \end{aligned}$$

(i.e., there is one fewer cycle of maximal length, and one more cycle of length one less than this) such that

$$(3.5) \quad \eta'(\alpha, \beta) \geq \eta'(\alpha', \beta').$$

Proof. Note that since $\alpha, \beta \neq e$, $a_r, b_s > 2$, so n is not a fixed point. Since $\eta'(a, b) = \eta(\mathfrak{s}(a), \mathfrak{s}(b))$, we may assume for simplicity that $\alpha = s(\alpha)$ and $\beta = s(\beta)$.

Suppose that

$$\begin{aligned} \alpha &= \mathfrak{s}(\alpha) = \alpha_2(u \dots n-2 \ n-1 \ n), \\ \beta &= \mathfrak{s}(\beta) = \beta_2(v \dots n-2 \ n-1 \ n), \end{aligned}$$

where $u = n - a_r, v = n - b_s$, and α_2 fixes u, \dots, n and β_2 fixes v, \dots, n .

Now take $\sigma \in S_n$ with $(n)\sigma = n$ and $(n-1)\sigma = n-1$. Then

$$\begin{aligned} (\alpha^{-1})^\sigma \beta &= (\alpha_2^{-1})^\sigma \left(n \ n-1 \ \sigma(n-2) \ \dots \ \sigma(u) \right) \beta_2 \left(v \ \dots \ n-1 \ n \right) \\ &= (n)(\alpha_2^{-1})^\sigma \left(n-1 \ \sigma(n-2) \ \dots \ \sigma(u) \right) \beta_2 \left(v \ \dots \ n-1 \right). \end{aligned}$$

So setting

$$\begin{aligned} \alpha' &= \alpha_2 \left(u \ \dots \ n-2 \ n-1 \right) \\ \beta' &= \beta_2 \left(v \ \dots \ n-2 \ n-1 \right), \end{aligned}$$

we have

$$(\alpha^{-1})^\sigma \beta = ((\alpha')^{-1})^\sigma (\beta')^{-1}.$$

Note that since $\mathfrak{s}(\alpha) = \alpha$ and $\mathfrak{s}(\beta) = \beta$, we also have $\mathfrak{s}(\alpha') = \alpha'$ and $\mathfrak{s}(\beta') = \beta'$ (as elements of S_{n-1}). So

$$\begin{aligned} \eta'(\alpha', \beta') &= \#\{[(\alpha')^{-1}]^\sigma \beta' | \sigma \in S_{n-1}, \sigma \text{ fixes } n-1\} \\ &= \#\{[(\alpha)^{-1}]^\sigma \beta | \sigma \in S_n, \sigma \text{ fixes } n-1 \text{ and } n\} \\ &\leq \#\{[(\alpha)^{-1}]^\sigma \beta | \sigma \in S_n, \sigma \text{ fixes } n\} \\ &= \eta'(\alpha, \beta). \end{aligned}$$

Thus the inequality (3.5) holds. \square

Lemma 16. *For $\alpha = (1 \ 2 \ 3 \ 4)$ and $\beta \in S_n$ with $n \geq 8$, and $\beta(1) \neq 1$, we have*

$$\eta'(\alpha, \beta) \geq 5$$

Proof. Suppose that β consists of at least 3 cycles, at least 2 of which are nontrivial. Suppose with the cycle lengths written in increasing order, with longest cycles of lengths b_1, b_2, b_3 , we have

$$[\beta] = [S, b_1, b_2, b_3],$$

where S is some sequence of integers, and $b_2, b_3 \geq 2$. We can assume $\beta = \mathfrak{s}(\beta)$, and the cycles of the lengths b_1, b_2, b_3 start $(m_1 \ \cdots)$, $(m_2 \ m_3 \ \cdots)$, $(m_4 \ \cdots n)$, so

$$\beta = \mathfrak{s}(\beta) = \beta_2(m_1 \ \cdots)(m_2 \ m_3 \ \cdots)(m_4 \ \cdots n),$$

where β_2 fixes $1, \dots, m_1-1$, and $m_1 = n - b_1 - b_2 - b_3$, $m_2 = n - b_2 - b_3$, $m_3 = m_2 + 1$, and $m_4 = n - b_3$.

Then we have at least the following possible cycle types of elements of the form $(\mathfrak{s}(\alpha)^{-1})^\sigma \mathfrak{s}(\beta)$, with $\sigma(n) = n$:

$$\begin{aligned} [(n \ m_1 \ m_2 \ m_3) \cdot \beta_2(m_1 \ \cdots)(m_2 \ m_3 \ \cdots)(m_4 \ \cdots n)] &= b_1 + b_3 + 1, b_2 - 1, S \\ [(n \ m_1 \ m_2 \ m_4) \cdot \beta_2(m_1 \ \cdots)(m_2 \ m_3 \ \cdots)(m_4 \ \cdots n)] &= b_1 + b_2 + 1, b_3 - 1, S \\ [(n \ m_2 \ m_1 \ m_3) \cdot \beta_2(m_1 \ \cdots)(m_2 \ m_3 \ \cdots)(m_4 \ \cdots n)] &= b_3 + 1, b_1 + b_2 - 1, S \\ [(n \ m_3 \ m_1 \ m_2) \cdot \beta_2(m_1 \ \cdots)(m_2 \ m_3 \ \cdots)(m_4 \ \cdots n)] &= b_1 + 1, b_2 + b_3 - 1, S \\ [(n \ m_1 \ m_3 \ m_2) \cdot \beta_2(m_1 \ \cdots)(m_2 \ m_3 \ \cdots)(m_4 \ \cdots n)] &= b_1 + b_2 + b_3 - 1, 1, S \\ [(n \ m_2 \ m_4 \ m_3) \cdot \beta_2(m_1 \ \cdots)(m_2 \ m_3 \ \cdots)(m_4 \ \cdots n)] &= b_1, b_2 + b_3, S \\ [(n \ m_2 \ m_3 \ m_4) \cdot \beta_2(m_1 \ \cdots)(m_2 \ m_3 \ \cdots)(m_4 \ \cdots n)] &= 2, b_1, b_2 - 1, b_3 - 1, S \\ [(n \ m_4 \ m_2 \ m_3) \cdot \beta_2(m_1 \ \cdots)(m_2 \ m_3 \ \cdots)(m_4 \ \cdots n)] &= 1, b_1, b_2 - 1, b_3, S \\ [(n \ m_4 \ m_3 \ m_2) \cdot \beta_2(m_1 \ \cdots)(m_2 \ m_3 \ \cdots)(m_4 \ \cdots n)] &= 1, 1, b_1, b_2 + b_3 - 2, S \\ [(n \ m_3 \ m_2 \ m_4) \cdot \beta_2(m_1 \ \cdots)(m_2 \ m_3 \ \cdots)(m_4 \ \cdots n)] &= 1, b_1, b_2, b_3 - 1, S \end{aligned}$$

(Note that the cycle lengths in these sequences on the right hand sides are not in general in increasing numerical order.) In general, these will be distinct. In special cases some of these cycle types will be the same; but we can show that there are at least 4 different cycle types in all cases, and if there are only 4 cycle types, then we must have either $b_2 = b_3 = 2$ or $b_1 = 1, b_2 = 2, b_3 = 3$.

In these cases $b_2 = 2$ and $b_3 \leq 3$, so except possibly for one cycle of length 3, all cycles must have length at most 2 (since b_1, b_2, b_3 were maximal length cycles in β), and since $n \geq 8$, there are at least 4 cycles, so we can take

$$[\beta] = [R, b_4, b_1, b_2, b_3],$$

with b_2, b_2, b_3 as above, and R a sequence, and b_4 an integer, with $R, b_4 = S$. Then

$$[(n \ m_5 \ m_1 \ m_2) \cdot \beta_2(m_5 \ \cdots)(m_1 \ \cdots)(m_2 \ m_3 \ \cdots)(m_4 \ \cdots n)] = b_1 + b_2 + b_3 + b_4, R,$$

which is distinct from any of the above listed cycle types, since it has few terms in this sequence. Thus in this case $\eta'(\alpha, \beta) \geq 5$.

Now assume that β consists of either 2 nontrivial cycles, and no fixed points (since two nontrivial cycles and one fixed point is just the case of three cycles with $b_1 = 1$), or exactly two cycles, one of which may have length one. We write

$$\beta = \mathfrak{s}(\beta) = \beta_2(m_5 \cdots)(m_2 m_3 m_4 \cdots n),$$

with $m_2 = n - b_2$ and $m_5 = n - b_1 - b_2$, and with β having cycle structure S, b_1, b_2 . Since $n \geq 8$, at least one cycle has length at least 4, so, including one possible case where there is a cycle of length at least 5, we have cycle types

$$\begin{aligned} [(n m_2 m_3 m_4) \cdot \beta_2(m_5 \cdots)(n m_2 m_3 m_4 \cdots)] &= 2, b_1 - 2, b_2, S \\ [(n m_4 m_3 m_2) \cdot \beta_2(m_5 \cdots)(n m_2 m_3 m_4 \cdots)] &= 1, 1, 1, b_1 - 3, b_2, S \\ [(n m_2 m_4 m_3) \cdot \beta_2(m_5 \cdots)(n m_2 m_3 m_4 \cdots)] &= 1, b_1 - 1, b_2, S \\ [(n m_4 m_2 m_3) \cdot \beta_2(m_5 \cdots)(n m_2 m_3 m_4 \cdots)] &= 3, b_1 - 3, b_2, S \\ [(n m_2 m_3 m_5) \cdot \beta_2(m_5 \cdots)(n m_2 m_3 m_4 \cdots)] &= b_1 + b_2, S \\ [(n m_5 m_3 m_2) \cdot \beta_2(m_5 \cdots)(n m_2 m_3 m_4 \cdots)] &= 1, 1, b_1 + b_2 - 2, b_3, S \\ [(1 m_5 m_4 m_2) \cdot \beta_2(m_5 \cdots)(n m_2 m_3 m_4 m_5 \cdots)] &= 1, 1, 2, b_1 - 4, b_2, S \end{aligned}$$

Note that in the case $b_1 = 4$, the third and fourth line of the above table have the same cycle type, but altogether there are 5 different cycle types. If β consists of only one cycle, then β is a cycle of length n , then take the first four line and the last line of the above table to give five different possible cycle types of $\alpha^\sigma \beta$.

These cases cover all the possibilities, and so the result follows. \square

Proposition 17. *Suppose that $\alpha, \beta \in S_n$ for $n \geq 9$. Suppose $\eta(\alpha^{S_n} \beta^{S_n}) \leq 5$. Then one of α, β is conjugate to $()$, $(1\ 2)$, or $(1\ 2\ 3)$.*

Proof. We may assume neither of $\alpha, \beta = e$.

We can check the result for $n = 9$ with GAP. See Table 13 in the Appendix.

Suppose that $n > 9$, then by Proposition 15, we have, using the notation of that lemma,

$$\eta'(\alpha, \beta) \geq \eta'(\alpha', \beta').$$

If neither of α', β' is conjugate to $()$, $(1\ 2)$, $(1\ 2\ 3)$, then the result follows by induction. The cases where α' has one of these forms are:

$$\begin{aligned} \alpha &= (1\ 2) \\ \alpha &= (1\ 2\ 3) \\ \alpha &= (1\ 2\ 3\ 4), \end{aligned}$$

and similarly for β . The first two cases are contained in the statement of the result.

Suppose that $\alpha = (1\ 2\ 3\ 4)$, and β is not conjugate to $()$, $(1\ 2)$ or $(1\ 2\ 3)$. Then by Lemma 16 to show that $\eta(\alpha^{S_n} \beta^{S_n}) \geq 5$. \square

3.3. Cycle structures when $\eta \leq 5$.

Lemma 18. *Suppose that β is a transposition and*

$$[\alpha] = [\underbrace{1, \dots, 1}_{n_0}, \underbrace{a_1, \dots, a_1}_{n_1}, \underbrace{a_2, \dots, a_2}_{n_2}, \dots, \underbrace{a_r, \dots, a_r}_{n_r}],$$

where $n_i \geq 1$ for $1 \leq i \leq r$. Let $r' = r$ if $n_0 = 0$ and $r' = r + 1$ otherwise. Let r'' be the number of n_i , $0 \leq i \leq r$, which are at least 2. Then

$$\eta(\alpha^{S_n} \beta^{S_n}) = \sum_{i=0}^r \left\lfloor \frac{a_i}{2} \right\rfloor + \binom{r'}{2} + r''.$$

Proof. When computing $\eta(\alpha^{S_n} \beta^{S_n})$, we may just compute the number of possible cycle types in $\alpha^{S_n} \beta^\tau$ for some fixed τ , and so we may assume $\beta = (1\ 2)$. Suppose that for some σ we have $\alpha^\sigma = c_1 c_2 \cdots c_m$, where c_i are disjoint cycles, and $m = \sum_{i=0}^r n_i$. We may assume that c_1 involves 1, and either c_1 or c_2 involves 2.

In the first case, c_1 may be a cycle of length a_1, a_2, \dots, a_r . If c_1 has length a_i , then $[(1\ 2)c_1] = [s, a_i - s]$, where s depends on sigma, and has possible values $1 \leq s < a_i$, so there are $\lfloor \frac{a_i}{2} \rfloor$ possibilities for the set $\{s, a_i - s\}$. Now $\alpha^\sigma(1\ 2)$ has type $[\alpha^\sigma(1\ 2)] = T(i, s)$ where

$$T(i, s) := \underbrace{\{1, \dots, 1\}}_{n_0}, \underbrace{\{a_1, \dots, a_1\}}_{n_1}, \dots, \underbrace{\{a_i, \dots, a_i\}}_{n_i-1}, \underbrace{\{s, a_i - s, \dots, a_r, \dots, a_r\}}_{n_r}.$$

As a_i varies, taking values a_1, a_2, \dots, a_r , and as s varies in the range $1 \leq s \leq \lfloor \frac{a_i}{2} \rfloor$, we obtain $\sum_{i=1}^r \lfloor \frac{a_i}{2} \rfloor$ different possible types, which we claim are all distinct from each other. To see this, suppose that we have $T(i, s_1) = T(j, s_2)$. Then we have the following equality of multisets:

$$\{s_1, a_i - s_1, a_j\} = \{a_i, s_2, a_j - s_2\}.$$

We now use the fact that $1 \leq s_1 \leq \lfloor \frac{a_i}{2} \rfloor$ and $1 \leq s_2 \leq \lfloor \frac{a_j}{2} \rfloor$. So we can't have $s_1 = a_i$, and we can't have $a_i - s_1 = a_i$. Suppose $s_1 = s_2$. Then $a_i - s_1 = a_j - s_2$, so $a_i = a_j$. Suppose that $s_1 = a_j - s_2$. Then $a_i - s_1 = s_2$ and $a_j = a_i$, and $s_1 + s_2 = a_i$, which, since $1 \leq s_1, s_2 \leq \lfloor \frac{a_i}{2} \rfloor$, is only possible if $s_1 = s_2$. So, in all cases, $a_i = a_j$ and $s_1 = s_2$.

Now suppose that c_1 involves 1 and c_2 involves 2. There are two cases. Either c_1 and c_2 both have length a_i for some i , which is only possible if $n_i > 1$, or else c_1 and c_2 have lengths a_i, a_j with $i \neq j$. In the first case, there are r'' possible cycle types, and in the second case, $\binom{r'}{2}$ cycle types. These cycle types are all distinct from the types where c_1 involves both 1 and 2, since then $\alpha^\sigma(1\ 2)$ consisted of $m+1$ distinct cycles, whereas in these cases there are $m-1$ distinct cycles. The cycles in these cases are all distinct from each other, since if not, we would have one of the following equalities of multisets:

$$\begin{aligned} \{a_j, a_j, 2a_i\} &= \{a_i, a_i, 2a_j\} \\ \{a_j, 2a_i\} &= \{a_i, a_i + a_j\} \\ \{a_j, a_k, 2a_i\} &= \{a_i, a_i, a_j + a_k\} \\ \{a_j, a_i + a_k\} &= \{a_k, a_i + a_j\} \\ \{a_k, a_l, a_i + a_j\} &= \{a_i, a_j, a_k + a_l\}, \end{aligned}$$

where i, j, k, l are distinct. But a consideration of each case shows that none of these are possible, and so all cases are distinct, and the result follows. \square

Corollary 19. *If β is a transposition, $\alpha \neq e$, and $\eta(\alpha^{S_n} \beta^{S_n}) \leq 5$, then α has cycle type one of the following:*

$$\begin{array}{ll}
\{n\} & \text{for } 2 \leq n \leq 11 \\
\underbrace{\{i, \dots, i\}}_m & \text{for } 2 \leq i \leq 9, m \geq 2 \text{ where } n = mi \\
\{i, j\} & \text{for } 1 \leq i < j \leq 8, i \leq 4 \text{ and } n = i + j \leq 9 \\
\{3, 7\} & \text{and } n = 10 \\
\underbrace{\{i, \dots, i, j\}}_m & \text{for } n = im + j \leq 8, i \leq 3, m \geq 2 \text{ and } \{i, j\} \neq \{2, 6\} \\
\underbrace{\{1, \dots, 1\}}_{m_1} \underbrace{\{j, \dots, j\}}_{m_2} & \text{for } m_1, m_2 \geq 2, m_1 + jm_2 = n \leq 6, 2 \leq j \leq 5 \\
\underbrace{\{2, \dots, 2\}}_{m_1} \underbrace{\{3, \dots, 3\}}_{m_2} & \text{for } 2m_1 + 3m_2 = n \\
\{1, 2, 3\} & n = 6
\end{array}$$

(1s are not omitted from the cycle type.)

Proof. By Lemma 18, we must have $r' \leq 3$.

Suppose $r' = 1$. Then all cycles have the same length $a_1 \geq 2$. If $n_1 = 1$, then $r'' = 0$, and $\eta = \lfloor a_1/2 \rfloor$, so $2 \leq a_1 \leq 11$. This gives the first case in the above list. If $n_1 \geq 2$, $r'' = 1$, and $\eta = \lfloor a_1/2 \rfloor + 1$, so $2 \leq a_1 \leq 9$. This gives the second line.

Suppose $r' = 2$, so there are two distinct cycle lengths, say a_i, a_j , possibly $a_i = 1$. $\eta = \lfloor a_i/2 \rfloor + \lfloor a_j/2 \rfloor + 1 + r''$. This gives the next 5 cases.

If $r' = 3$, the only possible case is the last one listed. \square

Lemma 20. *Suppose that β is a three cycle, and*

$$[\alpha] = [\underbrace{1, \dots, 1}_{n_0}, \underbrace{a_1, \dots, a_1}_{n_1}, \underbrace{a_2, \dots, a_2}_{n_2}, \dots, \underbrace{a_r, \dots, a_r}_{n_r}],$$

where $n_i \geq 1$ for $1 \leq i \leq r$. Let r' and r'' be as in Lemma 18. Let r''' be the number of n_i , $0 \leq i \leq r$, which are at least 3. Then

$$\begin{aligned}
\eta(\alpha^{S_n} \beta^{S_n}) &\leq \sum_{i=0}^r P_3(a_i) + \binom{r'}{3} + r''' + (r' - 1) \sum \lfloor a_i/2 \rfloor + s_3 \\
&\quad + \sum_{1 \leq i \leq r, a_i \geq 2, n_i \geq 2} (a_i - 1),
\end{aligned}$$

where $P_3(a_i)$ is the number of partitions of a_i into three parts, and $s_3 = 1$ if any $a_i \geq 3$, and 0 otherwise.

Proof. We may assume that $\beta = (1 \ 2 \ 3)$. There are a number of possible cases, a few of which are: 1, 2, 3 are all involved in the same cycle c , of length a_i , of α^c , in which case $[c\beta]$ can, amongst other possibilities, be any partition of a_i into 3 parts. This gives the term $P_3(a_i)$. Another possibility in this case is that $[c\beta] = \{a_i\}$. This gives the term s_3 .

Another possibility is that each of 1, 2, 3 are in different cycles, of different lengths. This gives the term $\binom{r'}{3}$ in the sum. A third case is when 1, 2, 3 are all in different cycles, but these cycles have the same length. This gives the term r''' .

Another possible case is 1, 2, 3 are all in different cycles, and these have lengths a_i, a_i, a_j for $a_i \neq a_j$. This gives the term $(r' - 1) \sum [a_i/2]$.

If 1 and 2 are both in the same cycle c_1 , and 3 is in a different cycle, c_2 , but c_1 and c_2 both have length a_i , then the type of $c_1 c_2(1\ 2\ 3)$ is $\{a_i + u, v\}$, where $u + v = a_i$, $u, v \geq 2$. The term $\sum (a_i - 1)$ comes from this case.

These cases can all shown to have distinct cycle type, in a similar manner to the proof of Lemma 18. There are a number of other cases, but for the inequality, we do not need to consider them. \square

Remark 4. The first few terms in the increasing sequence $P_3(a_i)$, for $i = 1, 2, 3, \dots$ are 0, 0, 1, 1, 2, 3, 4, 5, 7. This is Sloane's sequence A069905 [5], where more terms and a formula can be found.

Corollary 21. *With β a three cycle and $\alpha \in S_n \setminus \{e\}$, if $n > 8$ and $\eta(\alpha^{S_n} \beta^{S_n}) \leq 5$, then α has one of the following cycle types:*

$$\underbrace{\{2, \dots, 2\}}_{n_1}, \underbrace{\{3, \dots, 3\}}_{n_1}, \underbrace{\{1, \dots, 1, 2\}}_{n_0}, \underbrace{\{1, \dots, 1, 2, 2\}}_{n_0}, \underbrace{\{1, \dots, 1, 3\}}_{n_0}, \underbrace{\{1, \dots, 1, 4\}}_{n_0},$$

where n_0, n_1 are integers, $n_1 \geq 3$, $n_0 \geq 1$. In these cases, the values of $\eta(\alpha^{S_n} \beta^{S_n})$ are as follows respectively: 2, 4, 3, 5, 5, 5.

Proof. We use the inequality in Lemma 20. By Remark 4, we must have $a_i \leq 8$. We must also have $r' \leq 4$, $r'' \leq 5$. Let $R(\alpha)$ be the expression on the right of the inequality in Lemma 20. First suppose that all n_i are 1 or 0. A computer search for α with types satisfying $R(\alpha) \leq 5$ gives the following possible cases for $[\alpha]$:

$$\{1, 3, 3\}, \{1, 1, 3, 3\}\{4\}, \{4, 4\}, \{5\}, \{6\}, \{7\}, \{1, 5\}, \{1, 1, 5\}, \{2, 3\}, \{2, 2, 3\}\{2, 4\},$$

$$\underbrace{\{2, \dots, 2\}}_{n_1}, \underbrace{\{3, \dots, 3\}}_{n_1}, \underbrace{\{1, \dots, 1, 2, \dots, 2\}}_{n_0}, \underbrace{\{1, \dots, 1, 3\}}_{n_0}, \underbrace{\{1, \dots, 1, 4\}}_{n_0},$$

where n_0, n_1 are arbitrary positive integers. These have been listed so that only those in the second line satisfy $n > 8$.

To obtain the final result, we go through the cases with $n > 8$ one at a time. It is simple to see that representatives of all possible conjugacy classes of the product of $\alpha^{S_n} \beta^{S_n}$ in these cases are given as follows:

For $[\alpha] = \underbrace{\{2, \dots, 2\}}_{n_1}$:

$$\begin{aligned} (1\ 2\ 3)(1\ 2)(3\ 4)(5\ 6)\alpha_1 &= (2\ 4\ 3)(5\ 6)\alpha_1 \\ (1\ 3\ 5)(1\ 2)(3\ 4)(5\ 6)\alpha_1 &= (1\ 4\ 3\ 6\ 5\ 2)\alpha_1 \end{aligned}$$

where α_1 is a product of disjoint transpositions, acting on $7, \dots, n$.

For $[\alpha] = \underbrace{\{3, \dots, 3\}}_{n_1}$:

$$\begin{aligned} (1\ 2\ 3)(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)\alpha_1 &= (1\ 3\ 2)(4\ 5\ 6)(7\ 8\ 9)\alpha_1 \\ (1\ 3\ 2)(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)\alpha_1 &= (1)(2)(3)(4\ 5\ 6)(7\ 8\ 9)\alpha_1 \\ (1\ 2\ 4)(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)\alpha_1 &= (1\ 3)(2\ 5\ 6\ 4)(7\ 8\ 9)\alpha_1 \\ (1\ 4\ 5)(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)\alpha_1 &= (4\ 6)(1\ 5\ 2\ 3)(7\ 8\ 9)\alpha_1 \\ (1\ 3\ 4)(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)\alpha_1 &= (1)(2\ 3\ 5\ 6\ 4)(7\ 8\ 9)\alpha_1 \end{aligned}$$

where α_1 is a product of disjoint three cycles acting on $10, \dots, n$

For $[\alpha] = \{\underbrace{1, \dots, 1}_{n_0}, \underbrace{2, \dots, 2}_{n_1}\}$:

$$\begin{aligned} (1\ 2\ 3)(4\ 5)\alpha_1 &= (1\ 2\ 3)(4\ 5)\alpha_1 \\ (1\ 2\ 3)(3\ 4)\alpha_2 &= (1\ 2\ 4\ 3)\alpha_2 \\ (1\ 2\ 3)(2\ 3)\alpha_3 &= (1\ 3)\alpha_3 \\ (1\ 2\ 3)(2\ 4)(3\ 5)\alpha_1 &= (1\ 4\ 2\ 5\ 3)\alpha_1 \\ (1\ 2\ 3)(1\ 2)(3\ 4)\alpha_2 &= (2\ 4\ 3)\alpha_2 \\ (1\ 2\ 3)(1\ 4)(2\ 5)(3\ 6)\alpha_0 &= (1\ 5\ 2\ 6\ 3\ 4)\alpha_0 \end{aligned}$$

where $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ have order 2, and α_i fixes $1, 2, \dots, 6 - i$. The cases where α permutes either 2, 4 or more than 6 elements need to be distinguished from each other to determine the exact value of η in each case. For α a transposition, only the first three possibilities occur. For α a product of two disjoint transpositions, the first 5 cases occur. If α is a product of three or more disjoint transpositions, then all 6 cases occur.

For $[\alpha] = \{\underbrace{1, \dots, 1}_{n_0}, 3\}$:

$$\begin{aligned} (1\ 2\ 3)(4\ 5\ 6) &= (1\ 2\ 3)(4\ 5\ 6) \\ (1\ 2\ 3)(3\ 4\ 5) &= (1\ 2\ 4\ 5\ 3) \\ (1\ 2\ 3)(2\ 3\ 4) &= (1\ 3)(2\ 4) \\ (1\ 2\ 3)(3\ 2\ 4) &= (1\ 4\ 3) \\ (1\ 2\ 3)(1\ 2\ 3) &= (1\ 3\ 2) \\ (1\ 2\ 3)(1\ 3\ 2) &= () \end{aligned}$$

For $[\alpha] = \{\underbrace{1, \dots, 1}_{n_0}, 4\}$:

$$\begin{aligned} (1\ 2\ 3)(4\ 5\ 6\ 7) &= (1\ 2\ 3)(4\ 5\ 6\ 7) \\ (1\ 2\ 3)(3\ 4\ 5\ 6) &= (1\ 2\ 4\ 5\ 6\ 3) \\ (1\ 2\ 3)(2\ 3\ 4\ 5) &= (1\ 3)(2\ 4\ 5) \\ (1\ 2\ 3)(3\ 4\ 2\ 5) &= (2\ 4)(1\ 5\ 3) \\ (1\ 2\ 3)(3\ 2\ 4\ 5) &= (1\ 4\ 5\ 3) \\ (1\ 2\ 3)(1\ 2\ 3\ 4) &= (1\ 3\ 2\ 4) \\ (1\ 2\ 3)(1\ 3\ 2\ 4) &= (1\ 4) \end{aligned}$$

□

Proof Of Theorem 1. The minimum value of η in A_n is at most 5, since we always have $\eta((1\ 2\ 3)^{A_n}(1\ 2\ 3)^{A_n}) = 5$.

For $6 \leq n \leq 8$, the result follows from direct computation by GAP [3], as can be seen in the tables in the Appendix.

Suppose $n \geq 9$. From Proposition 8, Proposition 17, and Corollary 21, we see that the only possible cases with $\eta(\alpha^{A_n}\beta^{A_n}) < 5$ for $\alpha, \beta \in A_n \setminus \{e\}$ are those given in the statement of the theorem. □

4. APPENDIX

In this Appendix, we give tables of η and η' for A_n and S_n for $3 \leq n \leq 9$, as well as a table of minimal values of η up to $n = 12$. first we describe the GAP code used to produce these tables, which then follow.

4.1. GAP code. Given two elements i and j of a group G , the following code computes up to n representatives of the classes appearing in the set $j^G i^G$. Note that it suffices to only check products of the form bi , for $b \in j^G$, since any product $j^h i^g$ is conjugate to $j^{hg^{-1}} i$.

```

conjproduct:=function(G,i,j,n)
  local b,c,cj,k,rl,rlen,ok;
  rl:=[];
  rlen:=0;
  cj:=ConjugacyClass(G,j);
  for b in cj do
    if rlen=n then break; fi; #if found enough, stop
    ok:=true;
    k:=1;
    c:=i*b;
    while ok and k<=rlen do #compare against found classes
      if IsConjugate(G,rl[k][2],c) then ok:=false; fi;
      k:=k+1;
    od;
    if ok then #if new class, add to list
      Add(rl,[b,c]);
      rlen:=rlen+1;
    fi;
  od;
  return rl;
end;
    
```

Suppose that G has k conjugacy classes, cl_1, \dots, cl_k , and suppose that $n = m$, if $m \geq 1$, and otherwise, $n = k$. (For large n , it can take quite a bit of time to compute all values of η for S_n and A_n : we can save time, if desired, by setting an upper bound m for η .) The following code then uses the previous code to generate

- up to n representatives of the classes appearing in the set $cl_j cl_i$, for each pair $j \leq i$, and
- pairs g, h of representatives, listed by $\min(\eta(h^G g^G), n)$.

```

allconjproducts:=function(G,outputfile,m)

  local cl,ch,g,h,i,j,k,n,clist,clen,num,pair;

  cl:=ConjugacyClasses(G);
  k:=Length(cl);
  if m<1 then n:=k; else n:=m; fi; #find at most n classes
  num:=[];
  for i in [1..n] do
    Add(num,[]);
  od;

  AppendTo(outputfile,"\r\n\r\n",G,"\r\n\r\n");
  for i in [1..k] do
    g := Representative(cl[i]);
    for j in [1..i] do
      h:=Representative(cl[j]);
      if h=() then #trivial h means eta = 1
        AppendTo(outputfile,"[,g,]*[,h,]: 1 \r\n",
          "[ ,[h,g], " ] \r\n\r\n");
        Add(num[1],[g,h]);
      else
        clist:=conjproduct(G,g,h,n);
        clen:=Length(clist);
        AppendTo(outputfile,"[,g,]*[,h,]: ",clen,"\r\n",
          clist,"\r\n\r\n");
        Add(num[clen],[g,h]);
      fi;
    od;
  od;

  for i in [1..n] do #construct list of pairs, ordered by eta
    AppendTo(outputfile,"\r\n eta=",i,"\r\n");
    for pair in num[i] do
      AppendTo(outputfile,pair,"\r\n");
    od;
  od;

  return;

end;

```

For example,

```
allconjproducts(AlternatingGroup(3),"A3.txt",0);
```

yields a text file with the following contents:

```
AlternatingGroup( [ 1 .. 3 ] )
```

```
[( )]*[( )]: 1  
[ [ ( ), ( ) ] ]
```

```
[(1,2,3)]*[( )]: 1  
[ [ ( ), (1,2,3) ] ]
```

```
[(1,2,3)]*[(1,2,3)]: 1  
[ [ (1,2,3), (1,3,2) ] ]
```

```
[(1,3,2)]*[( )]: 1  
[ [ ( ), (1,3,2) ] ]
```

```
[(1,3,2)]*[(1,2,3)]: 1  
[ [ (1,2,3), ( ) ] ]
```

```
[(1,3,2)]*[(1,3,2)]: 1  
[ [ (1,3,2), (1,2,3) ] ]
```

```
eta=1:  
[ ( ), ( ) ]  
[ (1,2,3), ( ) ]  
[ (1,2,3), (1,2,3) ]  
[ (1,3,2), ( ) ]  
[ (1,3,2), (1,2,3) ]  
[ (1,3,2), (1,3,2) ]
```

```
eta=2:
```

```
eta=3:
```

n	$\min \eta$	pairs where attained
3	1	$(1\ 2\ 3), (1\ 2)$
4	1	$(1\ 2\ 3), (1\ 2)(3\ 4)$
5	2	$(1\ 2\ 3\ 4\ 5), (1\ 2)$
6	2	$(1\ 2)(3\ 4)(5\ 6), (1\ 2)$ $(1\ 2\ 3), (1\ 2)(3\ 4)(5\ 6)$ $(1\ 2\ 3)(4\ 5\ 6), (1\ 2)$
7	3	$(1\ 2), (1\ 2)$ $(1\ 2)(3\ 4)(5\ 6), (1\ 2)$ $(1\ 2\ 3), (1\ 2)$ $(1\ 2\ 3)(4\ 5\ 6), (1\ 2)$ $(1\ 2\ 3\ 4\ 5\ 6\ 7), (1\ 2)$
8	2	$(1\ 2)(3\ 4)(5\ 6)(7\ 8), (1\ 2)$ $(1\ 2\ 3), (1\ 2)(3\ 4)(5\ 6)(7\ 8)$
9	2	$(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9), (1\ 2)$
10	2	$(1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10), (1\ 2)$ $(1\ 2\ 3), (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)$
11	3	$(1\ 2), (1\ 2)$ $(1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10), (1\ 2)$ $(1\ 2\ 3), (1\ 2)$
12	2	$(1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12), (1\ 2)$ $(1\ 2\ 3), (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12)$ $(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12), (1\ 2)$

TABLE 1. Minimum values of η for S_n

n	$\min \eta$	pairs where attained
3	1	$(1\ 2\ 3), (1\ 2\ 3)$ $(1\ 3\ 2), (1\ 2\ 3)$ $(1\ 3\ 2), (1\ 3\ 2)$
4	1	$(1\ 2\ 3), (1\ 2)(3\ 4)$ $(1\ 2\ 3), (1\ 2\ 3)$ $(1\ 2\ 4), (1\ 2)(3\ 4)$ $(1\ 2\ 4), (1\ 2\ 4)$
5	3	$(1\ 2\ 3\ 4\ 5), (1\ 2)(3\ 4)$ $(1\ 2\ 3\ 5\ 4), (1\ 2)(3\ 4)$
6	5	$(1\ 2\ 3), (1\ 2)(3\ 4)$ $(1\ 2\ 3)(4\ 5\ 6), (1\ 2)(3\ 4)$ $(1\ 2\ 3)(4\ 5\ 6), (1\ 2\ 3)$ $(1\ 2\ 3\ 4)(5\ 6), (1\ 2\ 3)$ $(1\ 2\ 3\ 4)(5\ 6), (1\ 2\ 3)(4\ 5\ 6)$
7	5	$(1\ 2\ 3), (1\ 2)(3\ 4)$ $(1\ 2\ 3), (1\ 2\ 3)$
8	2	$(1\ 2\ 3), (1\ 2)(3\ 4)(5\ 6)(7\ 8)$
9	4	$(1\ 2\ 3), (1\ 2)(3\ 4)(5\ 6)(7\ 8)$
10	5	$(1\ 2\ 3), (1\ 2)(3\ 4)$ $(1\ 2\ 3), (1\ 2)(3\ 4)(5\ 6)(7\ 8)$ $(1\ 2\ 3), (1\ 2\ 3)$
11	5	$(1\ 2\ 3), (1\ 2)(3\ 4)$ $(1\ 2\ 3), (1\ 2\ 3)$
12	2	$(1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12), (1\ 2\ 3)$

 TABLE 2. Minimum values of η for A_n

$\eta(\alpha^{S_3}\beta^{S_3})$	$\bigcirc \begin{smallmatrix} (1\ 2) \\ (1\ 2\ 3) \end{smallmatrix}$	$\eta(\alpha^{A_3}\beta^{A_3})$	$\bigcirc \begin{smallmatrix} (1\ 2\ 3) \\ (1\ 3\ 2) \end{smallmatrix}$
$()$	1 1 1	$()$	1 1 1
$(1\ 2)$	1 2 1	$(1\ 2\ 3)$	1 1 1
$(1\ 2\ 3)$	1 1 2	$(1\ 3\ 2)$	1 1 1

 TABLE 3. Values of η for S_3 and A_3

$\eta(\alpha^{S_4}\beta^{S_4})$	\emptyset (1 2) (1 2)(3 4) (1 2 3) (1 2 3 4)
\emptyset	1 1 1 1 1
(1 2)	1 3 2 2 2
(1 2)(3 4)	1 2 2 1 2
(1 2 3)	1 2 1 3 2
(1 2 3 4)	1 2 2 2 3

$\eta(\alpha^{A_4}\beta^{A_4})$	\emptyset (1 2)(3 4) (1 2 3) (1 2 4)
\emptyset	1 1 1 1
(1 2)(3 4)	1 2 1 1
(1 2 3)	1 1 1 2
(1 2 4)	1 1 2 1

TABLE 4. Values of η for S_4 and A_4

$\eta(\alpha^{S_5}\beta^{S_5})$	\emptyset (1 2) (1 2)(3 4) (1 2 3) (1 2 3)(4 5) (1 2 3 4) (1 2 3 4 5)
\emptyset	1 1 1 1 1 1 1
(1 2)	1 3 3 3 3 3 2
(1 2)(3 4)	1 3 4 3 3 3 3
(1 2 3)	1 3 3 4 3 3 3
(1 2 3)(4 5)	1 3 3 3 4 3 3
(1 2 3 4)	1 3 3 3 3 4 3
(1 2 3 4 5)	1 2 3 3 3 3 4

$\eta(\alpha^{A_5}\beta^{A_5})$	\emptyset (1 2)(3 4) (1 2 3) (1 2 3 4 5) (1 2 3 5 4)
\emptyset	1 1 1 1 1
(1 2)(3 4)	1 5 4 3 3
(1 2 3)	1 4 5 4 4
(1 2 3 4 5)	1 3 4 4 4
(1 2 3 5 4)	1 3 4 4 4

TABLE 5. Values of η for S_5 and A_5

$\eta(\alpha^{S_6}\beta^{S_6})$	\emptyset (1 2) (1 2)(3 4) (1 2)(3 4)(5 6) (1 2 3) (1 2 3)(4 5) (1 2 3)(4 5 6) (1 2 3 4) (1 2 3 4)(5 6) (1 2 3 4 5) (1 2 3 4 5 6)
\emptyset	1 1 1 1 1 1 1 1 1 1 1
(1 2)	1 3 4 2 3 5 2 4 4 3 3
(1 2)(3 4)	1 4 6 4 4 4 4 5 5 5 4
(1 2)(3 4)(5 6)	1 2 4 3 2 3 3 4 4 3 5
(1 2 3)	1 3 4 2 5 5 4 4 4 5 4
(1 2 3)(4 5)	1 5 4 3 5 6 4 5 5 5 5
(1 2 3)(4 5 6)	1 2 4 3 4 4 5 4 4 5 5
(1 2 3 4)	1 4 5 4 4 5 4 6 5 5 5
(1 2 3 4)(5 6)	1 4 5 4 4 5 4 5 6 5 5
(1 2 3 4 5)	1 3 5 3 5 5 5 5 5 6 5
(1 2 3 4 5 6)	1 3 4 5 4 5 5 5 5 5 6

TABLE 6. Values of η for S_6

$\eta(\alpha^{S_7}\beta^{S_7})$	$()$	$(1\ 2)$	$(1\ 2)(3\ 4)$	$(1\ 2)(3\ 4)(5\ 6)$	$(1\ 2\ 3)$	$(1\ 2\ 3)(4\ 5)$	$(1\ 2\ 3)(4\ 5)(6\ 7)$	$(1\ 2\ 3)(4\ 5\ 6)$	$(1\ 2\ 3\ 4)$	$(1\ 2\ 3\ 4)(5\ 6)$	$(1\ 2\ 3\ 4)(5\ 6\ 7)$	$(1\ 2\ 3\ 4\ 5)$	$(1\ 2\ 3\ 4\ 5)(6\ 7)$	$(1\ 2\ 3\ 4\ 5\ 6)$	$(1\ 2\ 3\ 4\ 5\ 6\ 7)$
$()$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$(1\ 2)$	1	3	4	3	3	6	4	3	4	6	4	4	4	4	3
$(1\ 2)(3\ 4)$	1	4	7	7	5	7	6	5	6	7	5	7	6	6	5
$(1\ 2)(3\ 4)(5\ 6)$	1	3	7	7	4	7	7	6	6	6	6	6	7	7	6
$(1\ 2\ 3)$	1	3	5	4	5	7	5	6	5	6	5	6	5	6	5
$(1\ 2\ 3)(4\ 5)$	1	6	7	7	7	8	7	7	7	7	7	7	7	7	6
$(1\ 2\ 3)(4\ 5)(6\ 7)$	1	4	6	7	5	7	8	6	5	7	7	6	7	6	7
$(1\ 2\ 3)(4\ 5\ 6)$	1	3	5	6	6	7	6	8	6	7	7	7	6	7	7
$(1\ 2\ 3\ 4)$	1	4	6	6	5	7	5	6	7	7	6	7	6	7	6
$(1\ 2\ 3\ 4)(5\ 6)$	1	6	7	6	6	7	7	7	7	8	7	7	7	7	7
$(1\ 2\ 3\ 4)(5\ 6\ 7)$	1	4	5	6	5	7	7	7	6	7	8	6	7	7	7
$(1\ 2\ 3\ 4\ 5)$	1	4	7	6	6	7	6	7	7	7	6	8	7	7	7
$(1\ 2\ 3\ 4\ 5)(6\ 7)$	1	4	6	7	5	7	7	6	6	7	7	7	8	7	7
$(1\ 2\ 3\ 4\ 5\ 6)$	1	4	6	7	6	7	6	7	7	7	7	7	7	8	7
$(1\ 2\ 3\ 4\ 5\ 6\ 7)$	1	3	5	6	5	6	7	7	6	7	7	7	7	7	8

TABLE 7. Values of η for S_7

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$\eta(\alpha^{S_8} \beta^{S_8})$	
$()$	$()$
$(1\ 2)$	$(1\ 2)$
$(1\ 2)(3\ 4)$	$(1\ 2)(3\ 4)$
$(1\ 2)(3\ 4)(5\ 6)$	$(1\ 2)(3\ 4)(5\ 6)$
$(1\ 2)(3\ 4)(5\ 6)(7\ 8)$	$(1\ 2)(3\ 4)(5\ 6)(7\ 8)$
$(1\ 2\ 3)$	$(1\ 2\ 3)$
$(1\ 2\ 3)(4\ 5)$	$(1\ 2\ 3)(4\ 5)$
$(1\ 2\ 3)(4\ 5)(6\ 7)$	$(1\ 2\ 3)(4\ 5)(6\ 7)$
$(1\ 2\ 3)(4\ 5\ 6)$	$(1\ 2\ 3)(4\ 5\ 6)$
$(1\ 2\ 3)(4\ 5\ 6)(7\ 8)$	$(1\ 2\ 3)(4\ 5\ 6)(7\ 8)$
$(1\ 2\ 3\ 4)$	$(1\ 2\ 3\ 4)$
$(1\ 2\ 3\ 4)(5\ 6)$	$(1\ 2\ 3\ 4)(5\ 6)$
$(1\ 2\ 3\ 4)(5\ 6)(7\ 8)$	$(1\ 2\ 3\ 4)(5\ 6)(7\ 8)$
$(1\ 2\ 3\ 4)(5\ 6\ 7)$	$(1\ 2\ 3\ 4)(5\ 6\ 7)$
$(1\ 2\ 3\ 4)(5\ 6\ 7\ 8)$	$(1\ 2\ 3\ 4)(5\ 6\ 7\ 8)$
$(1\ 2\ 3\ 4\ 5)$	$(1\ 2\ 3\ 4\ 5)$
$(1\ 2\ 3\ 4\ 5)(6\ 7)$	$(1\ 2\ 3\ 4\ 5)(6\ 7)$
$(1\ 2\ 3\ 4\ 5)(6\ 7\ 8)$	$(1\ 2\ 3\ 4\ 5)(6\ 7\ 8)$
$(1\ 2\ 3\ 4\ 5\ 6)$	$(1\ 2\ 3\ 4\ 5\ 6)$
$(1\ 2\ 3\ 4\ 5\ 6)(7\ 8)$	$(1\ 2\ 3\ 4\ 5\ 6)(7\ 8)$
$(1\ 2\ 3\ 4\ 5\ 6\ 7)$	$(1\ 2\ 3\ 4\ 5\ 6\ 7)$
$(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)$	$(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)$

TABLE 8. Values of η for S_8

$\eta(\alpha^{A_7} \beta^{A_7})$	$()$ $(1\ 2)(3\ 4)$ $(1\ 2\ 3)$ $(1\ 2\ 3)(4\ 5)(6\ 7)$ $(1\ 2\ 3)(4\ 5\ 6)$ $(1\ 2\ 3\ 4)(5\ 6)$ $(1\ 2\ 3\ 4\ 5)$ $(1\ 2\ 3\ 4\ 5\ 6\ 7)$ $(1\ 2\ 3\ 4\ 5\ 7\ 6)$
$()$	1 1 1 1 1 1 1 1 1
$(1\ 2)(3\ 4)$	1 7 5 7 6 8 8 6 6
$(1\ 2\ 3)$	1 5 5 6 7 7 7 6 6
$(1\ 2\ 3)(4\ 5)(6\ 7)$	1 7 6 9 7 8 7 8 8
$(1\ 2\ 3)(4\ 5\ 6)$	1 6 7 7 9 8 8 8 8
$(1\ 2\ 3\ 4)(5\ 6)$	1 8 7 8 8 9 8 8 8
$(1\ 2\ 3\ 4\ 5)$	1 8 7 7 8 8 9 8 8
$(1\ 2\ 3\ 4\ 5\ 6\ 7)$	1 6 6 8 8 8 8 8 9
$(1\ 2\ 3\ 4\ 5\ 7\ 6)$	1 6 6 8 8 8 8 9 8

TABLE 11. Values of η for A_7

$\eta(\alpha^{A_8} \beta^{A_8})$	$()$ $(1\ 2)(3\ 4)$ $(1\ 2)(3\ 4)(5\ 6)(7\ 8)$ $(1\ 2\ 3)$ $(1\ 2\ 3)(4\ 5)(6\ 7)$ $(1\ 2\ 3)(4\ 5\ 6)$ $(1\ 2\ 3\ 4)(5\ 6)$ $(1\ 2\ 3\ 4)(5\ 6\ 7\ 8)$ $(1\ 2\ 3\ 4\ 5)$ $(1\ 2\ 3\ 4\ 5)(6\ 7\ 8)$ $(1\ 2\ 3\ 4\ 5)(6\ 8\ 7)$ $(1\ 2\ 3\ 4\ 5\ 6)(7\ 8)$ $(1\ 2\ 3\ 4\ 5\ 6\ 7)$ $(1\ 2\ 3\ 4\ 5\ 6\ 8)$
$()$	1 1 1 1 1 1 1 1 1 1 1 1 1 1
$(1\ 2)(3\ 4)$	1 8 5 5 12 11 13 10 9 9 9 11 10 10
$(1\ 2)(3\ 4)(5\ 6)(7\ 8)$	1 5 5 2 9 7 11 8 7 7 7 12 10 10
$(1\ 2\ 3)$	1 5 2 5 10 9 9 7 9 9 9 9 10 10
$(1\ 2\ 3)(4\ 5)(6\ 7)$	1 12 9 10 14 12 13 11 12 13 13 13 13 13
$(1\ 2\ 3)(4\ 5\ 6)$	1 11 7 9 12 14 13 12 12 12 12 12 13 13
$(1\ 2\ 3\ 4)(5\ 6)$	1 13 11 9 13 13 14 13 13 12 12 13 13 13
$(1\ 2\ 3\ 4)(5\ 6\ 7\ 8)$	1 10 8 7 11 12 13 14 10 12 12 13 13 13
$(1\ 2\ 3\ 4\ 5)$	1 9 7 9 12 12 13 10 13 12 12 12 13 13
$(1\ 2\ 3\ 4\ 5)(6\ 7\ 8)$	1 9 7 9 13 12 12 12 12 13 13 13 13 13
$(1\ 2\ 3\ 4\ 5)(6\ 8\ 7)$	1 9 7 9 13 12 12 12 12 13 13 13 13 13
$(1\ 2\ 3\ 4\ 5\ 6)(7\ 8)$	1 11 12 9 13 12 13 13 12 13 13 14 13 13
$(1\ 2\ 3\ 4\ 5\ 6\ 7)$	1 10 10 10 13 13 13 13 13 13 13 13 13 14
$(1\ 2\ 3\ 4\ 5\ 6\ 8)$	1 10 10 10 13 13 13 13 13 13 13 13 14 13

TABLE 12. Values of η for A_8

$\eta(\alpha^{A_9}\beta^{A_9})$	\emptyset $(1\ 2)(3\ 4)$ $(1\ 2)(3\ 4)(5\ 6)(7\ 8)$ $(1\ 2\ 3)$ $(1\ 2\ 3)(4\ 5)(6\ 7)$ $(1\ 2\ 3)(4\ 5\ 6)$ $(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$ $(1\ 2\ 3\ 4)(5\ 6)$ $(1\ 2\ 3\ 4)(5\ 6\ 7)(8\ 9)$ $(1\ 2\ 3\ 4)(5\ 6\ 7\ 8)$ $(1\ 2\ 3\ 4\ 5)$ $(1\ 2\ 3\ 4\ 5)(6\ 7)(8\ 9)$ $(1\ 2\ 3\ 4\ 5)(6\ 7\ 8)$ $(1\ 2\ 3\ 4\ 5)(6\ 7\ 9)$ $(1\ 2\ 3\ 4\ 5\ 6)(7\ 8)$ $(1\ 2\ 3\ 4\ 5\ 6\ 7)$ $(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)$ $(1\ 2\ 3\ 4\ 5\ 6\ 7\ 9\ 8)$
\emptyset	1
$(1\ 2)(3\ 4)$	1
$(1\ 2)(3\ 4)(5\ 6)(7\ 8)$	1
$(1\ 2\ 3)$	1
$(1\ 2\ 3)(4\ 5)(6\ 7)$	1
$(1\ 2\ 3)(4\ 5\ 6)$	1
$(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$	1
$(1\ 2\ 3\ 4)(5\ 6)$	1
$(1\ 2\ 3\ 4)(5\ 6\ 7)(8\ 9)$	1
$(1\ 2\ 3\ 4)(5\ 6\ 7\ 8)$	1
$(1\ 2\ 3\ 4\ 5)$	1
$(1\ 2\ 3\ 4\ 5)(6\ 7)(8\ 9)$	1
$(1\ 2\ 3\ 4\ 5)(6\ 7\ 8)$	1
$(1\ 2\ 3\ 4\ 5)(6\ 7\ 9)$	1
$(1\ 2\ 3\ 4\ 5\ 6)(7\ 8)$	1
$(1\ 2\ 3\ 4\ 5\ 6\ 7)$	1
$(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)$	1
$(1\ 2\ 3\ 4\ 5\ 6\ 7\ 9\ 8)$	1

 TABLE 13. Values of η for A_9

$\eta'(\alpha, \beta)$	\emptyset $(1\ 2)(3\ 4)$ $(1\ 2\ 3)$ $(1\ 2\ 3)(4\ 5\ 6)$ $(1\ 2\ 3\ 4)(5\ 6)$ $(1\ 2\ 3\ 4\ 5)$
\emptyset	1
$(1\ 2)(3\ 4)$	1
$(1\ 2\ 3)$	1
$(1\ 2\ 3)(4\ 5\ 6)$	1
$(1\ 2\ 3\ 4)(5\ 6)$	1
$(1\ 2\ 3\ 4\ 5)$	1

 TABLE 14. Values of η' for even permutations of S_6

$\eta'(\alpha, \beta)$	$()$ $(1\ 2)(3\ 4)$ $(1\ 2\ 3)$ $(1\ 2\ 3)(4\ 5)(6\ 7)$ $(1\ 2\ 3)(4\ 5\ 6)$ $(1\ 2\ 3\ 4)(5\ 6)$ $(1\ 2\ 3\ 4\ 5)$ $(1\ 2\ 3\ 4\ 5\ 6\ 7)$
$()$	1
$(1\ 2)(3\ 4)$	1
$(1\ 2\ 3)$	1
$(1\ 2\ 3)(4\ 5)(6\ 7)$	1
$(1\ 2\ 3)(4\ 5\ 6)$	1
$(1\ 2\ 3\ 4)(5\ 6)$	1
$(1\ 2\ 3\ 4\ 5)$	1
$(1\ 2\ 3\ 4\ 5\ 6\ 7)$	1

TABLE 15. Values of η' for even permutations of S_7

$\eta'(\alpha, \beta)$	$()$ $(1\ 2)(3\ 4)$ $(1\ 2)(3\ 4)(5\ 6)(7\ 8)$ $(1\ 2\ 3)$ $(1\ 2\ 3)(4\ 5)(6\ 7)$ $(1\ 2\ 3)(4\ 5\ 6)$ $(1\ 2\ 3\ 4)(5\ 6)$ $(1\ 2\ 3\ 4)(5\ 6\ 7\ 8)$ $(1\ 2\ 3\ 4\ 5)$ $(1\ 2\ 3\ 4\ 5)(6\ 7\ 8)$ $(1\ 2\ 3\ 4\ 5\ 6)(7\ 8)$ $(1\ 2\ 3\ 4\ 5\ 6\ 7)$
$()$	1
$(1\ 2)(3\ 4)$	1
$(1\ 2)(3\ 4)(5\ 6)(7\ 8)$	1
$(1\ 2\ 3)$	1
$(1\ 2\ 3)(4\ 5)(6\ 7)$	1
$(1\ 2\ 3)(4\ 5\ 6)$	1
$(1\ 2\ 3\ 4)(5\ 6)$	1
$(1\ 2\ 3\ 4)(5\ 6\ 7\ 8)$	1
$(1\ 2\ 3\ 4\ 5)$	1
$(1\ 2\ 3\ 4\ 5)(6\ 7\ 8)$	1
$(1\ 2\ 3\ 4\ 5\ 6)(7\ 8)$	1
$(1\ 2\ 3\ 4\ 5\ 6\ 7)$	1

TABLE 16. Values of η' for even permutations of S_8

$\eta'(\alpha, \beta)$	$()$	$(1\ 2)(3\ 4)$	$(1\ 2)(3\ 4)(5\ 6)(7\ 8)$	$(1\ 2\ 3)$	$(1\ 2\ 3)(4\ 5)(6\ 7)$	$(1\ 2\ 3)(4\ 5\ 6)$	$(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$	$(1\ 2\ 3\ 4)(5\ 6)$	$(1\ 2\ 3\ 4)(5\ 6\ 7)(8\ 9)$	$(1\ 2\ 3\ 4)(5\ 6\ 7\ 8)$	$(1\ 2\ 3\ 4\ 5)$	$(1\ 2\ 3\ 4\ 5)(6\ 7)(8\ 9)$	$(1\ 2\ 3\ 4\ 5)(6\ 7\ 8)$	$(1\ 2\ 3\ 4\ 5\ 6)(7\ 8)$	$(1\ 2\ 3\ 4\ 5\ 6\ 7)$	$(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)$
$()$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$(1\ 2)(3\ 4)$	1	7	8	4	11	10	6	11	10	11	8	9	10	13	11	8
$(1\ 2)(3\ 4)(5\ 6)(7\ 8)$	1	8	12	4	14	11	7	13	13	12	9	14	12	14	13	13
$(1\ 2\ 3)$	1	4	4	4	7	7	5	7	7	7	6	6	9	10	9	8
$(1\ 2\ 3)(4\ 5)(6\ 7)$	1	11	14	7	16	14	13	15	14	14	13	15	14	15	15	13
$(1\ 2\ 3)(4\ 5\ 6)$	1	10	11	7	14	16	13	15	14	14	13	12	15	14	15	13
$(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$	1	6	7	5	13	13	13	11	14	12	10	12	15	14	13	15
$(1\ 2\ 3\ 4)(5\ 6)$	1	11	13	7	15	15	11	16	15	15	14	14	14	15	15	13
$(1\ 2\ 3\ 4)(5\ 6\ 7)(8\ 9)$	1	10	13	7	14	14	14	15	16	15	12	15	15	15	14	15
$(1\ 2\ 3\ 4)(5\ 6\ 7\ 8)$	1	11	12	7	14	14	12	15	15	16	12	14	15	15	15	15
$(1\ 2\ 3\ 4\ 5)$	1	8	9	6	13	13	10	14	12	12	12	12	14	14	15	13
$(1\ 2\ 3\ 4\ 5)(6\ 7)(8\ 9)$	1	9	14	6	15	12	12	14	15	14	12	16	14	15	14	15
$(1\ 2\ 3\ 4\ 5)(6\ 7\ 8)$	1	10	12	9	14	15	15	14	15	15	14	14	16	15	15	15
$(1\ 2\ 3\ 4\ 5\ 6)(7\ 8)$	1	13	14	10	15	14	14	15	15	15	14	15	15	16	15	15
$(1\ 2\ 3\ 4\ 5\ 6\ 7)$	1	11	13	9	15	15	13	15	14	15	15	14	15	15	16	15
$(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)$	1	8	13	8	13	13	15	13	15	15	13	15	15	15	15	16

TABLE 17. Values of η' for even permutations of S_9